

Linear Classifiers

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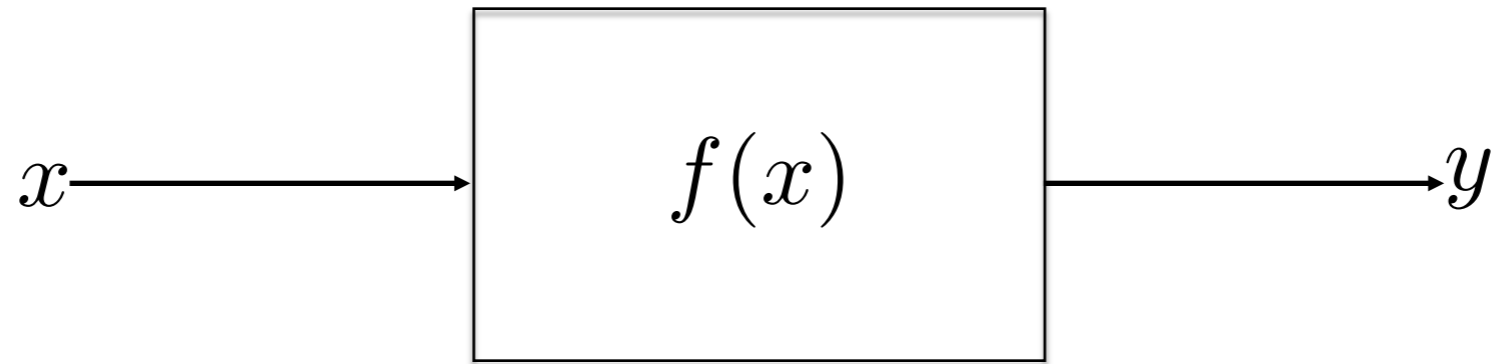
Overview

- Philosophical questions
- Derivatives: What are they good for?
- Linear regression
- Multiple linear regression
- Logistic regression

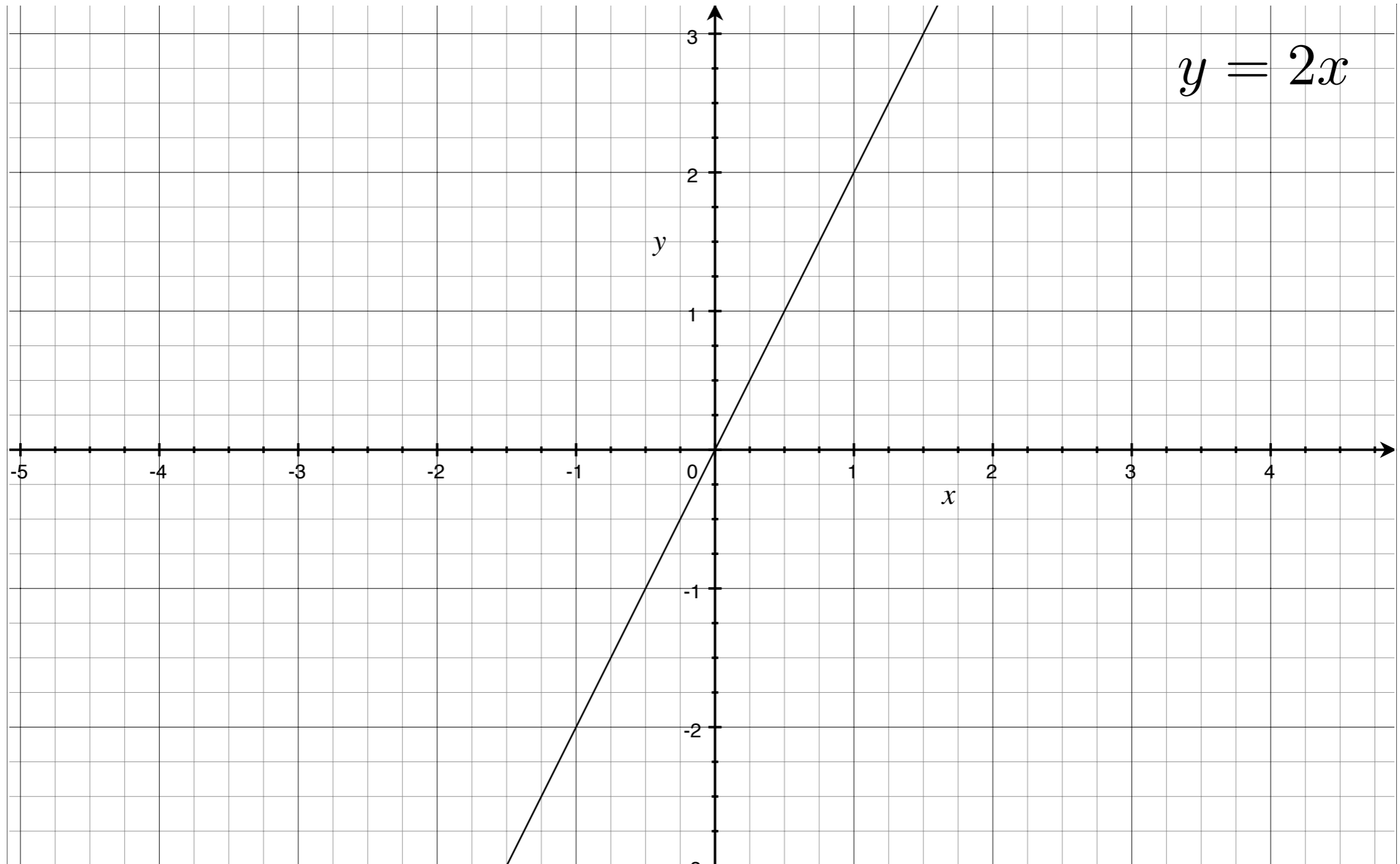
Philosophical Questions

- What would you do if ...
- What does this have to do with linear classifiers?

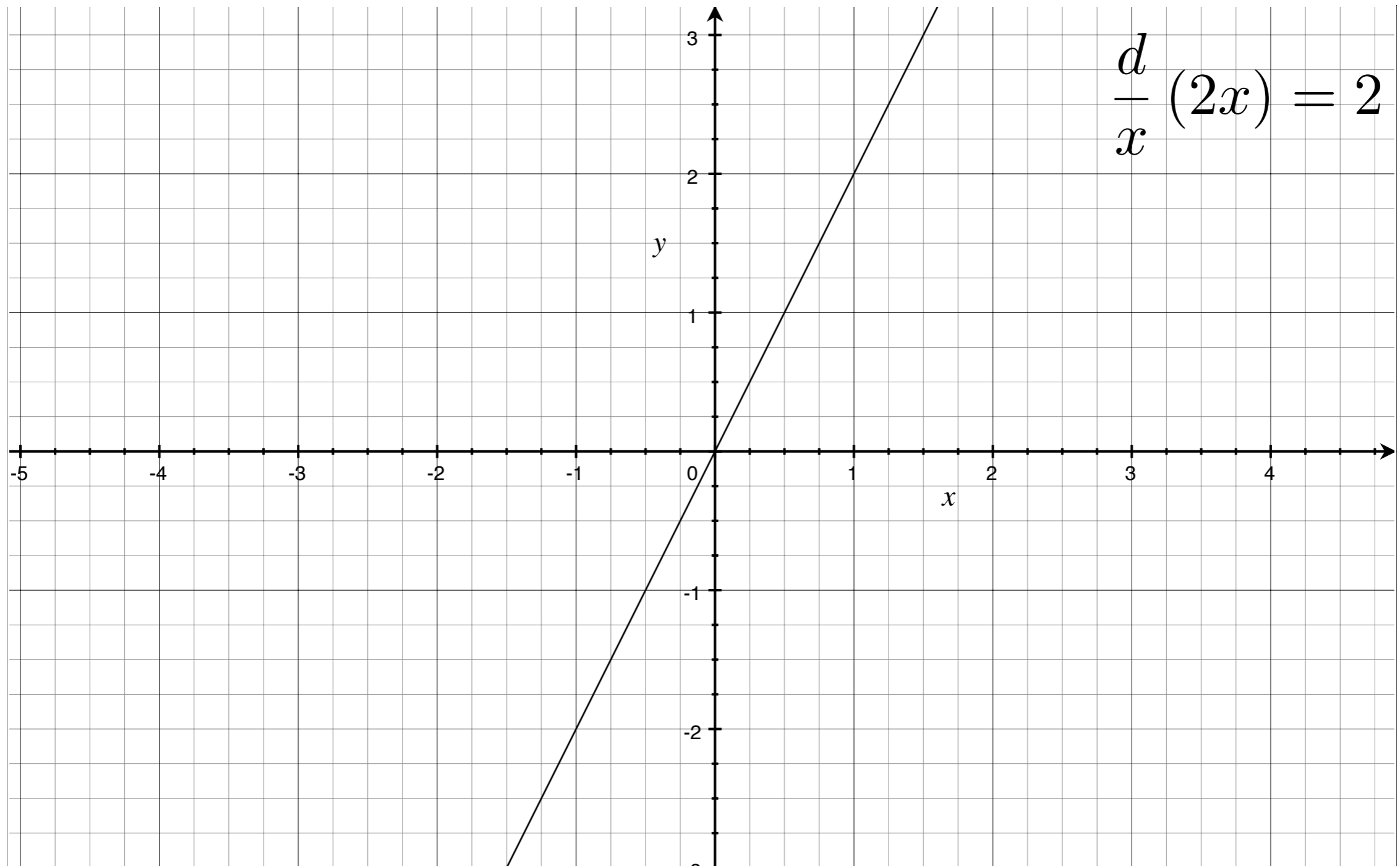
Functions



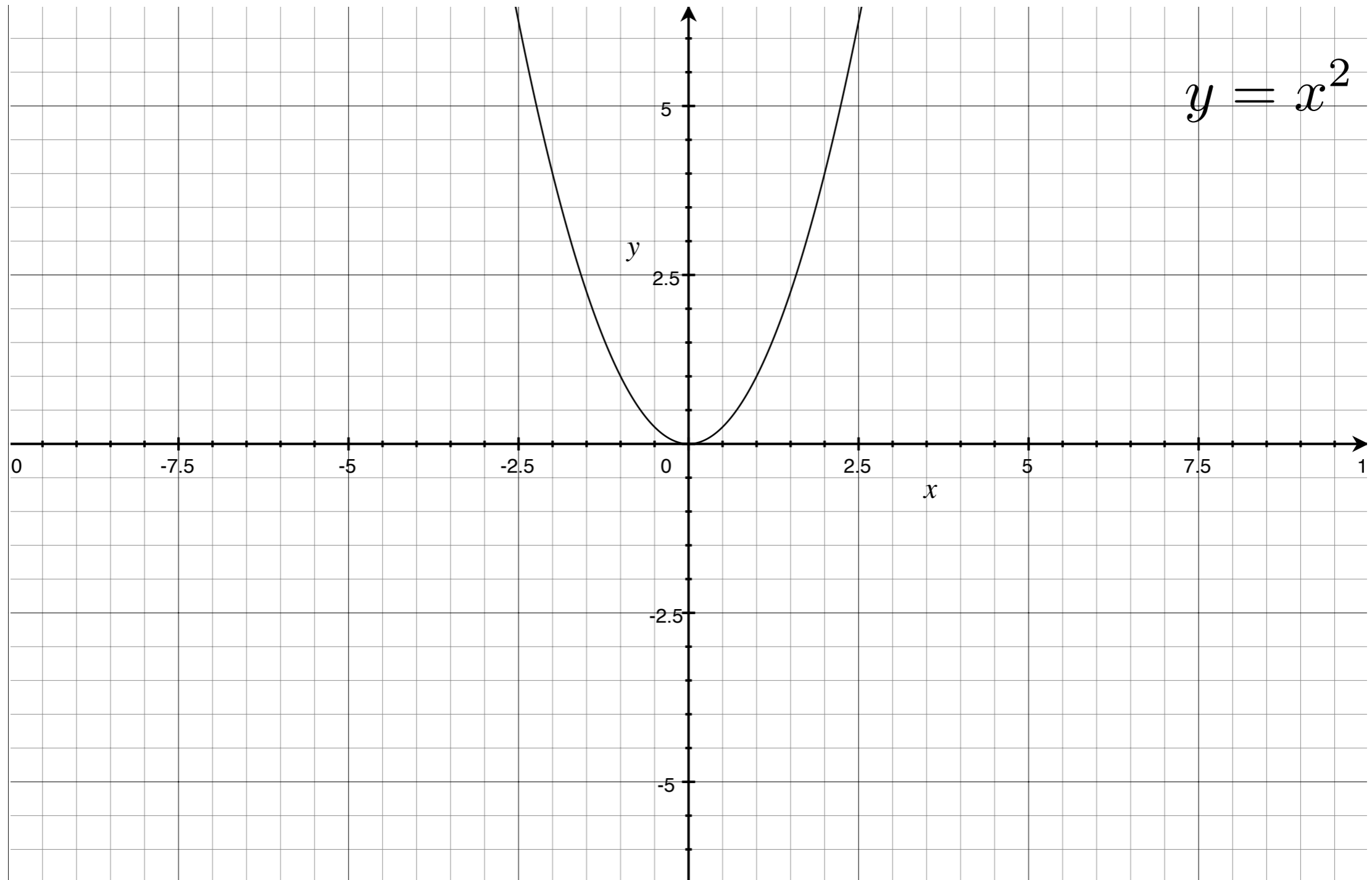
Derivatives



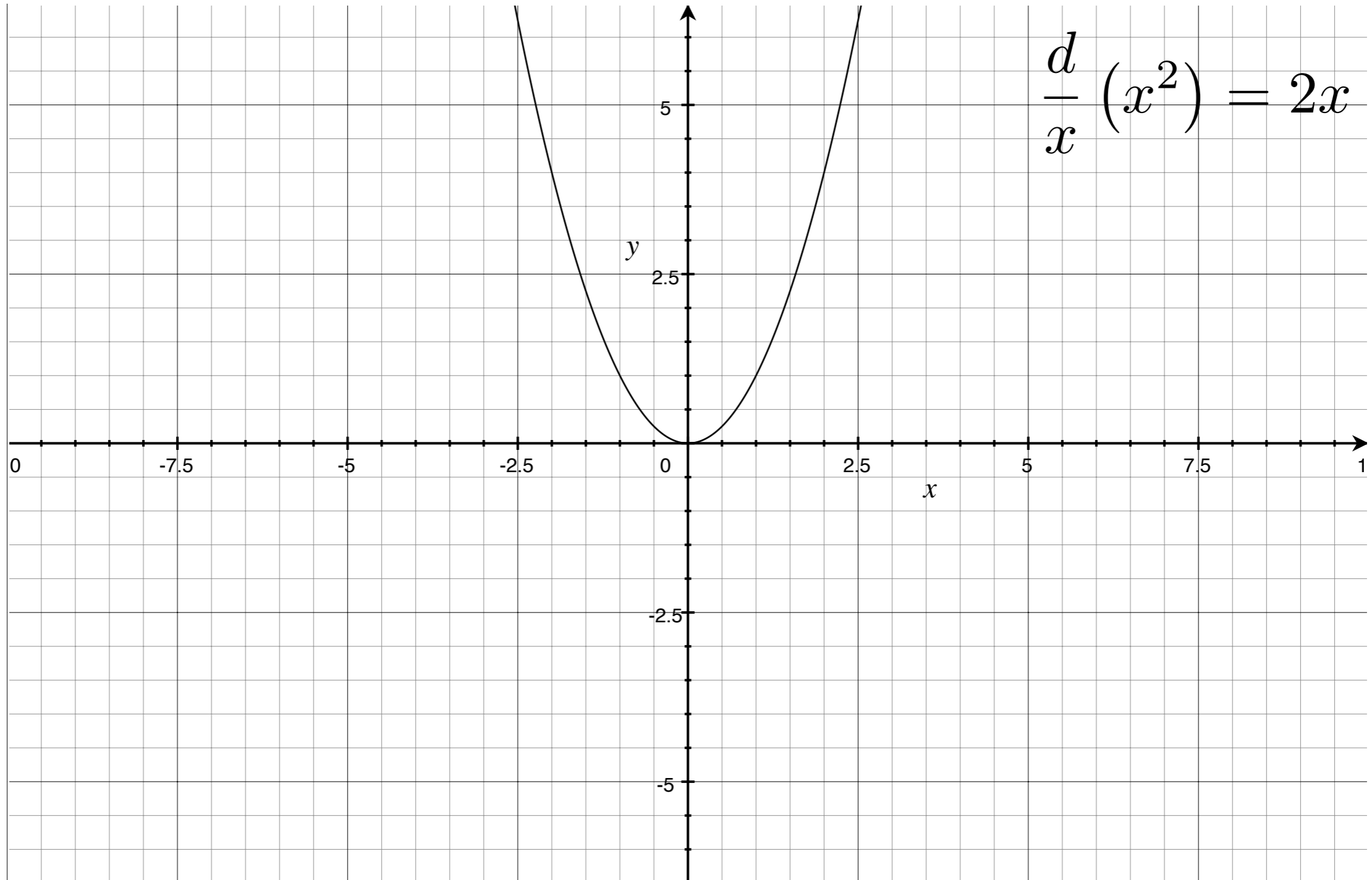
Derivatives



Derivatives



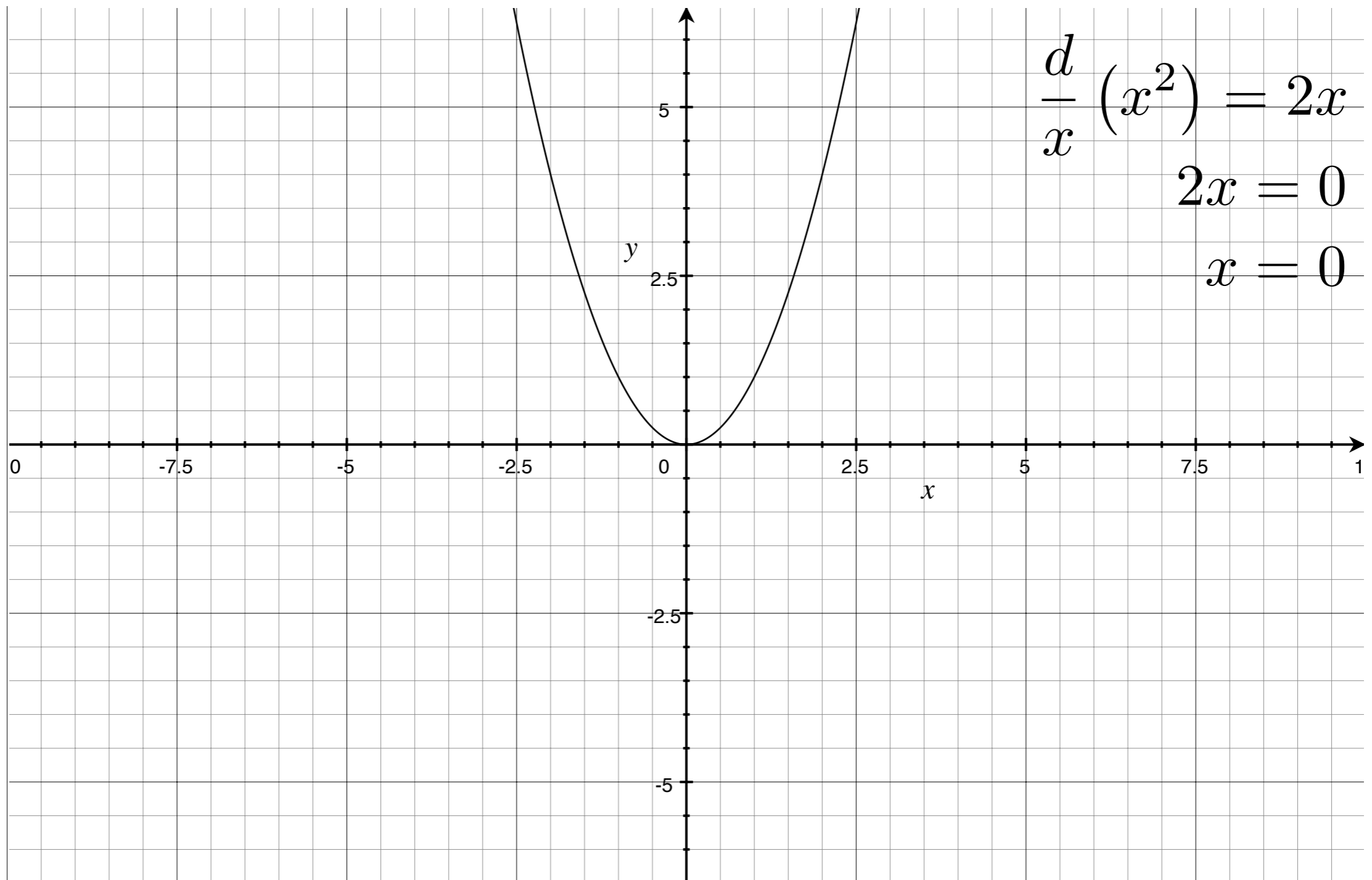
Derivatives



Derivatives: What are they good for?

- The derivative of $f(x)$ outputs the slope of $f(x)$ for a particular value of x
- A point of which the slope is zero is a point at which $f(x)$ is at its highest or lowest value.
- What does this have to do with machine learning?

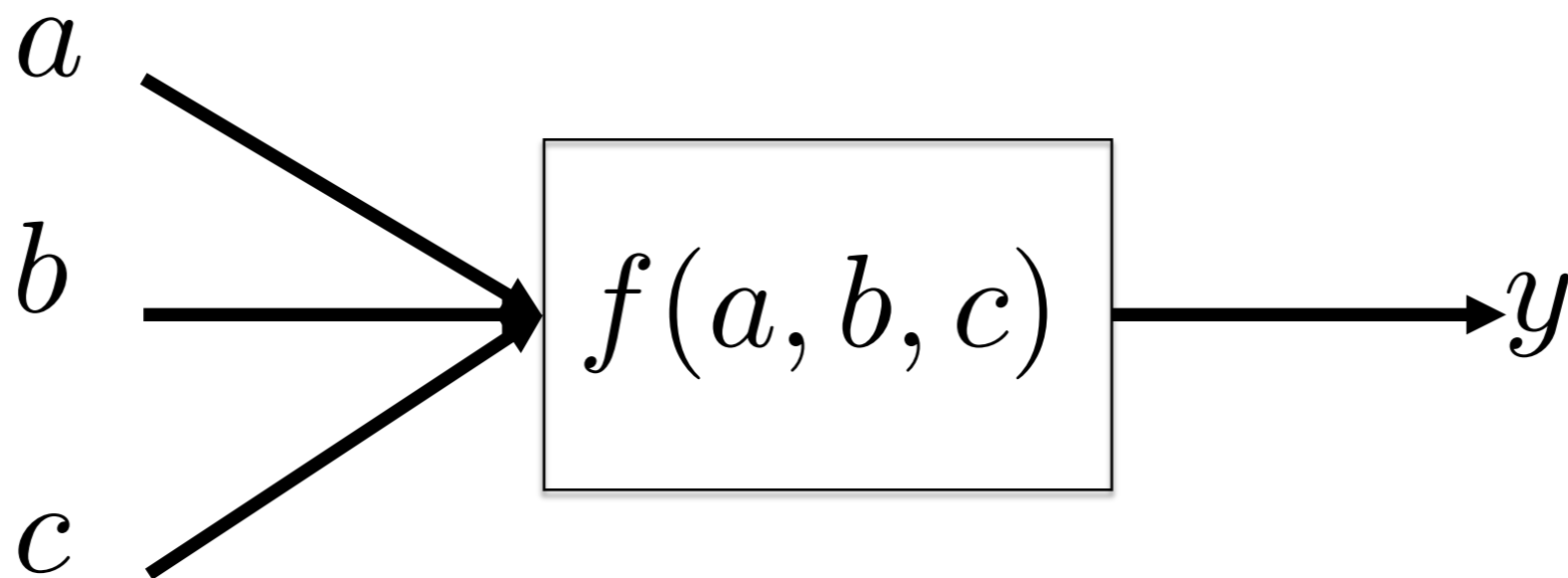
Derivatives



$$\begin{aligned} \frac{d}{dx} (x^2) &= 2x \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

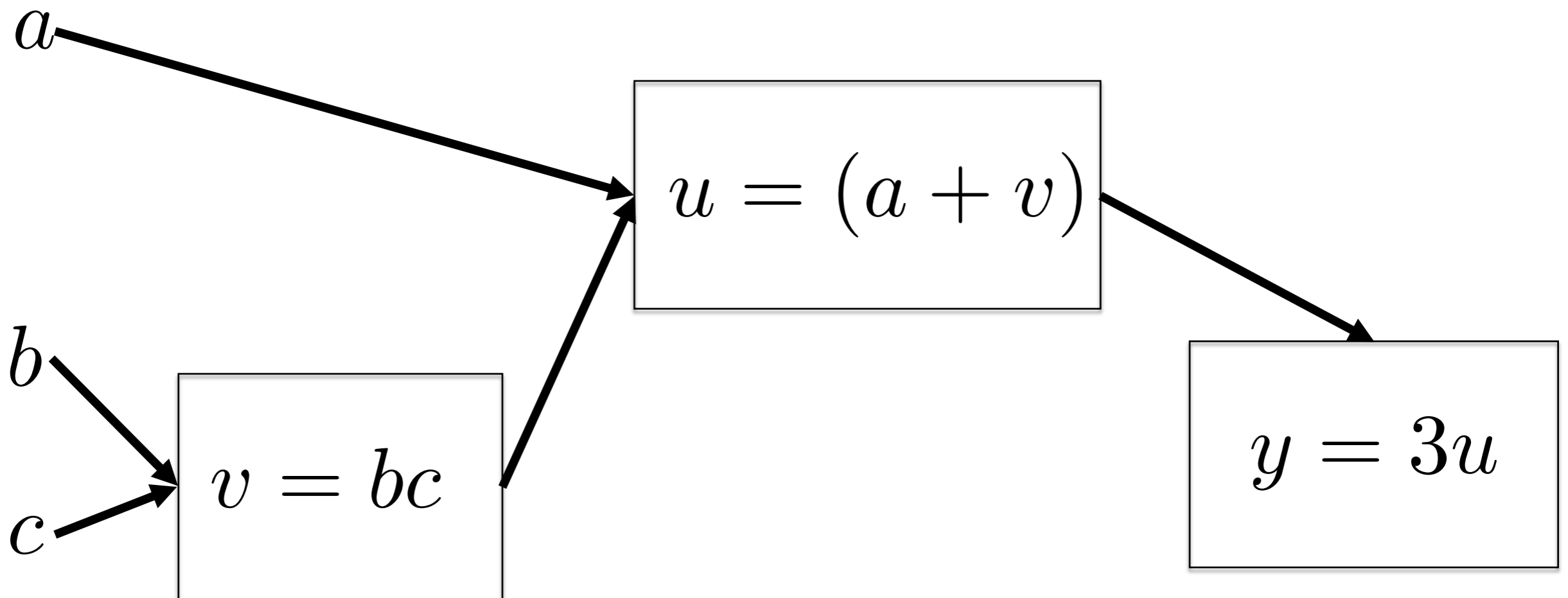
Computation Graphs

$$y = 3(a + bc)$$



Computation Graphs

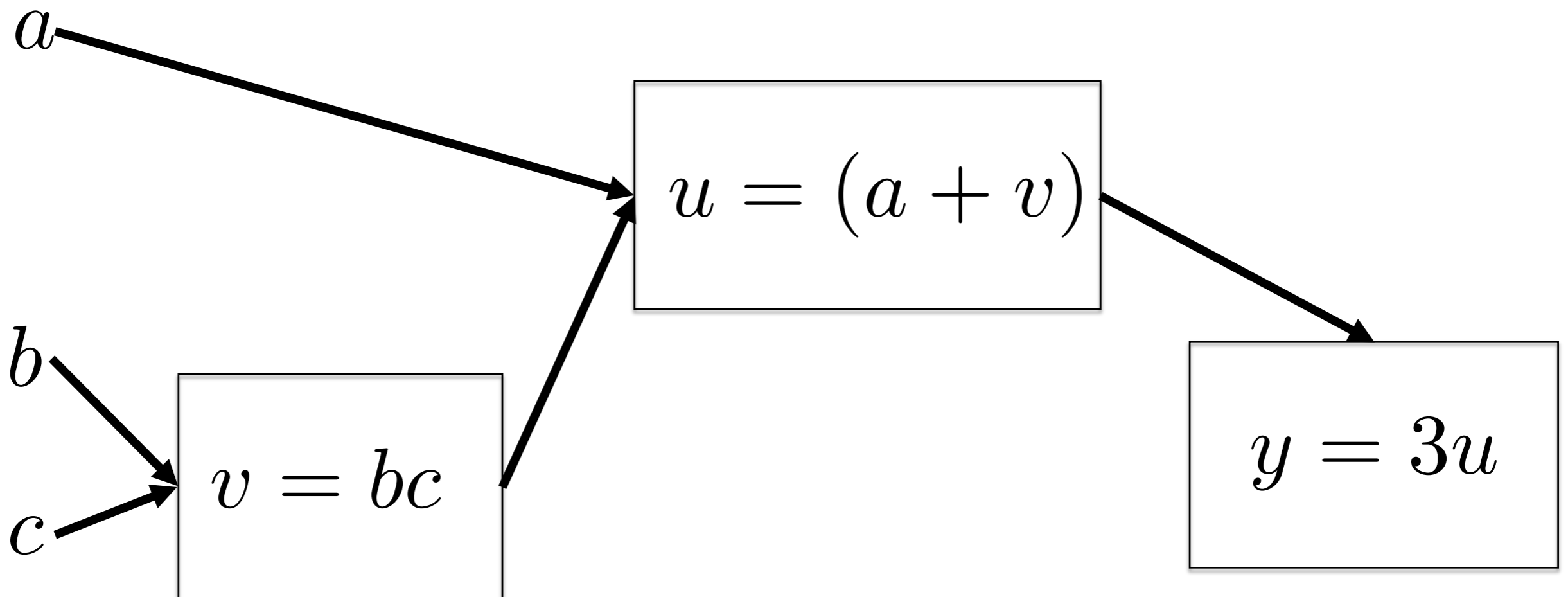
$$y = 3(a + bc)$$



Derivatives: Chain Rule

$$y = 3(a + bc)$$

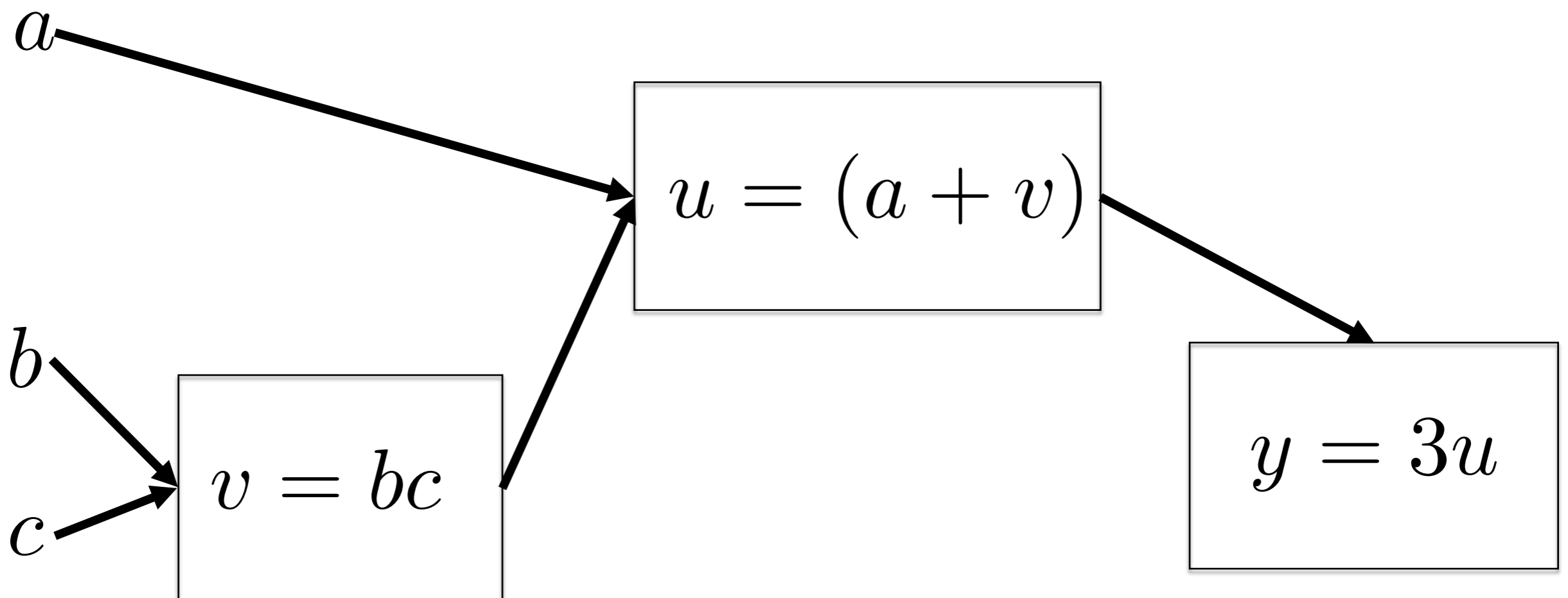
$$\frac{dy}{dc} = \frac{dv}{dc} \times \frac{du}{dv} \times \frac{dy}{du}$$



Derivatives: Chain Rule

$$y = 3(a + bc)$$

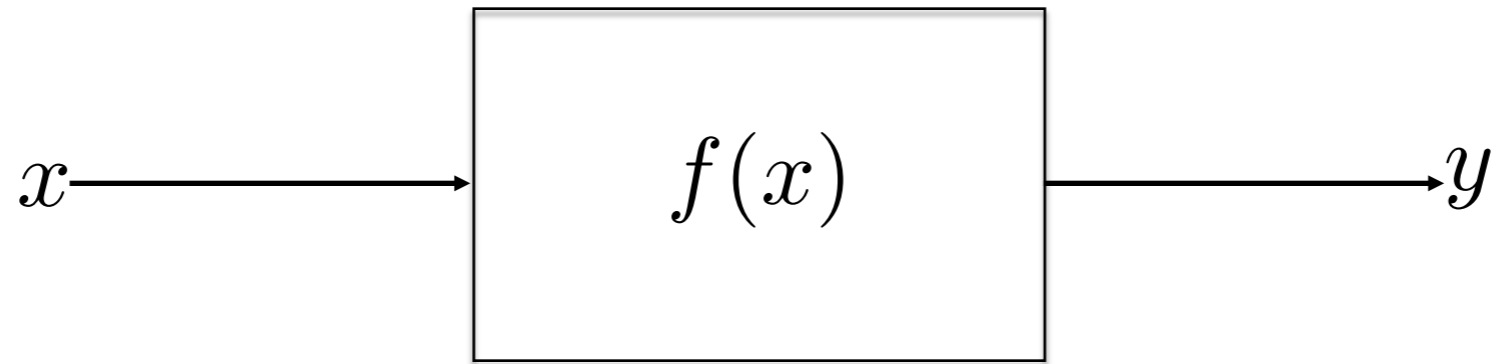
$$\frac{dy}{dc} = b \times 1 \times 3 = 3b$$



Overview

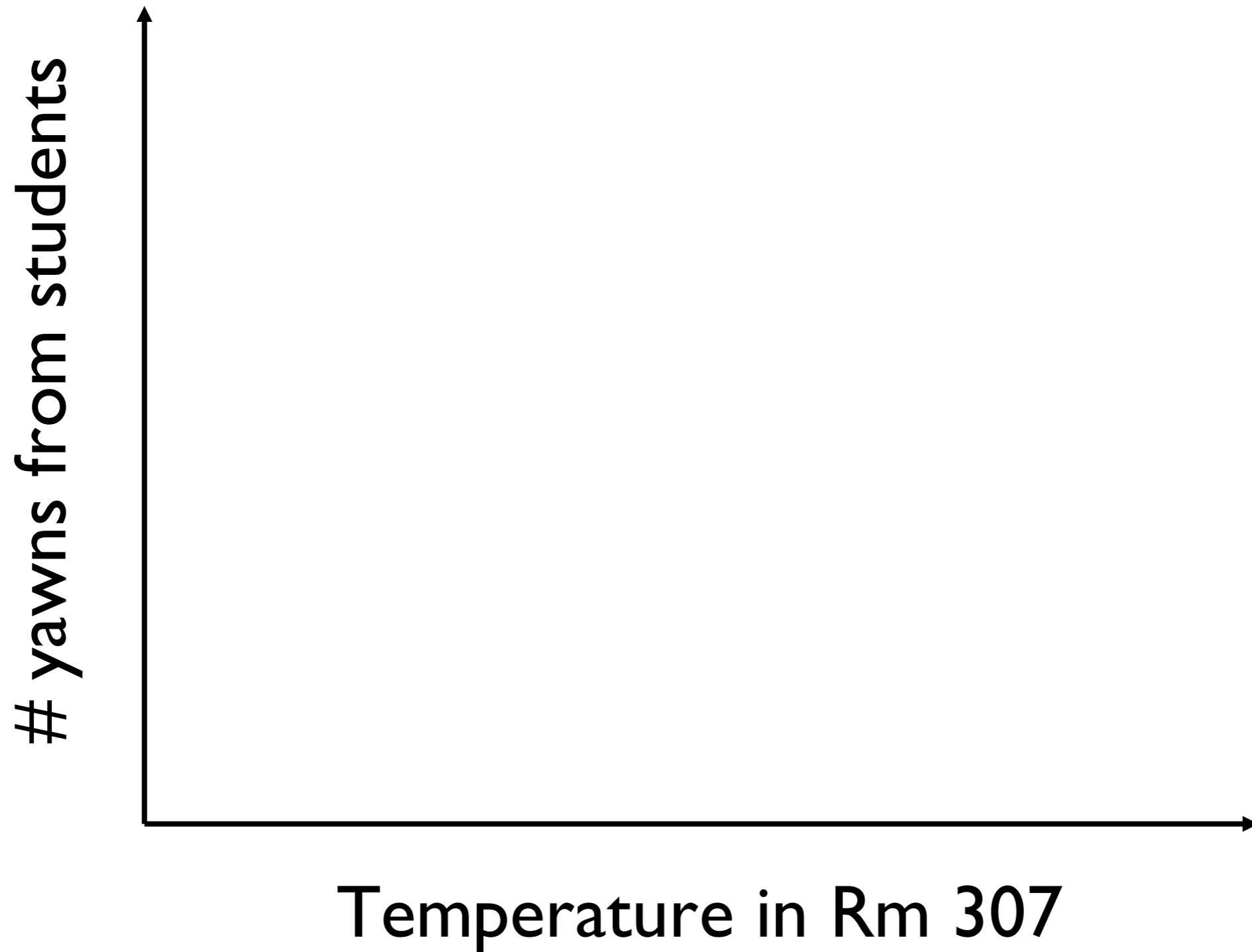
- Philosophical questions
- Derivatives: What are they good for?
- **Linear regression**
- Multiple linear regression
- Logistic regression

Linear Regression

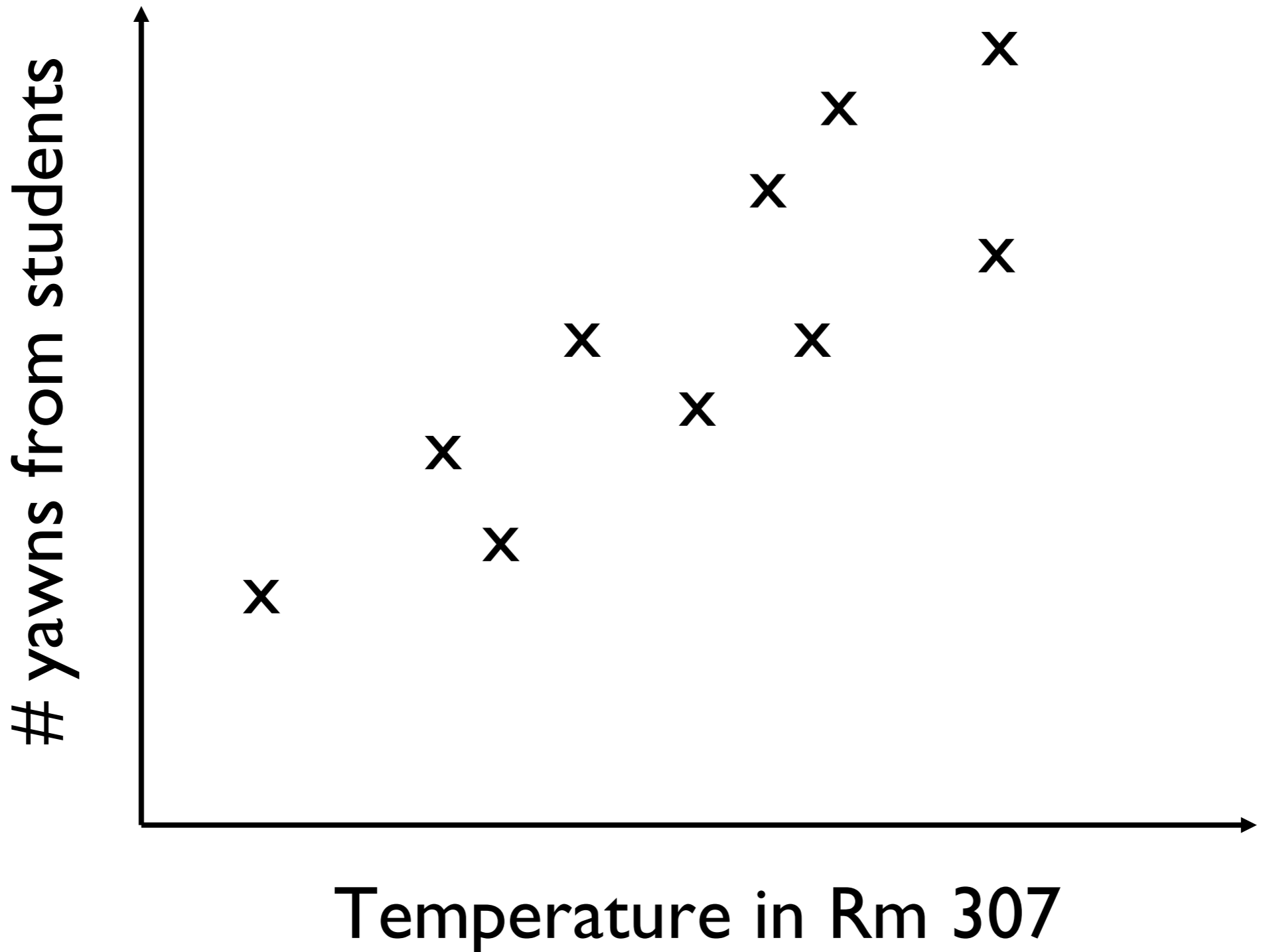


$$y = wx + b$$

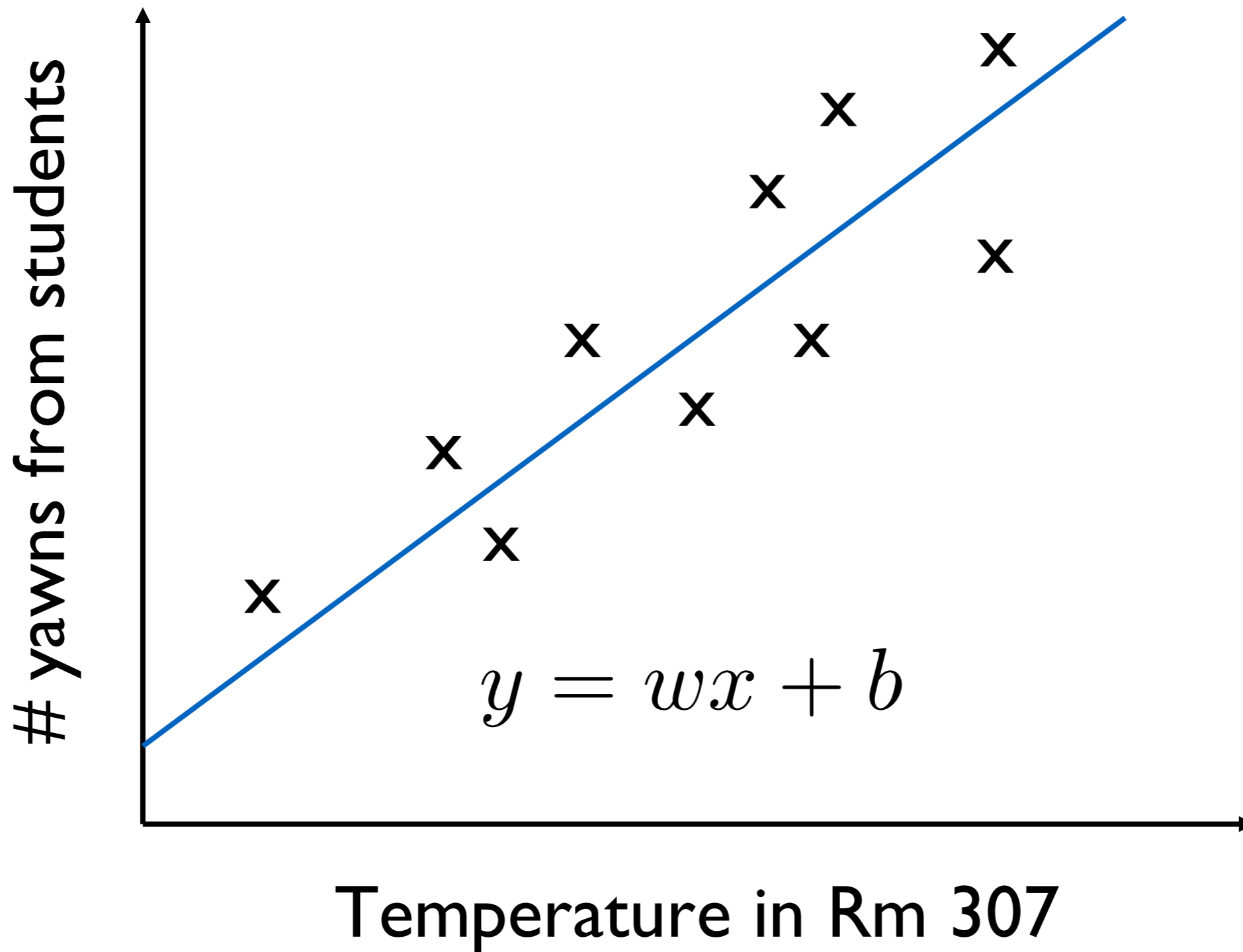
Linear Regression



Linear Regression



Linear Regression



Linear Regression: Training

$$y = wx + b$$

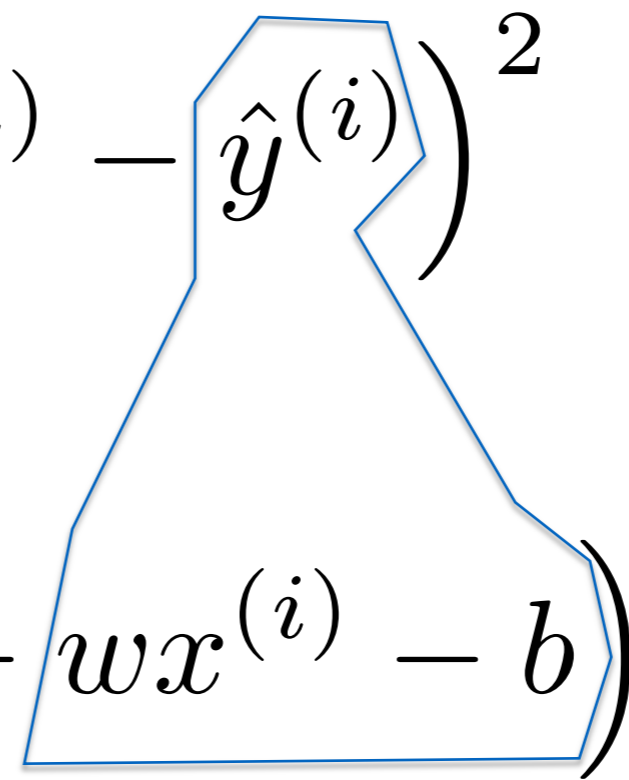
- **Input:** set of m training examples (x, y)
- Find the value of w and b that minimize the error:

$$\sum_{i=1}^m \left(y^{(i)} - \hat{y}^{(i)} \right)^2$$

Linear Regression: Training

$$y = wx + b$$

- Find the value of w and b that minimize the error:

$$\sum_{i=1}^m \left(y^{(i)} - \hat{y}^{(i)} \right)^2$$
$$\sum_{i=1}^m \left(y^{(i)} - wx^{(i)} - b \right)^2$$


Linear Regression: Training

$$y = wx + b$$

- Find the value of w and b that minimize the error:

$$\sum_{i=1}^m \left(y^{(i)} - wx^{(i)} - b \right)^2$$

- Take the derivative with respect to w , set it equal to 0, and solve for w .
- Take the derivative with respect to b , set it equal to 0, and solve for b .

Linear Regression: Training

- Find the value of w and b that minimize the error:

$$w = \frac{\frac{1}{m} \sum_{i=1}^m (x^{(i)} - \bar{x}) (y^{(i)} - \bar{y})}{\sum_{i=1}^m (x^{(i)} - \bar{x})^2}$$

$$b = \bar{y} - w\bar{x}$$

Linear Regression: Training

- Find the value of w and b that minimize the error:

$$w = \frac{\frac{1}{m} \sum_{i=1}^m (x^{(i)} - \bar{x}) (y^{(i)} - \bar{y})}{\sum_{i=1}^m (x^{(i)} - \bar{x})^2}$$

Always
positive!

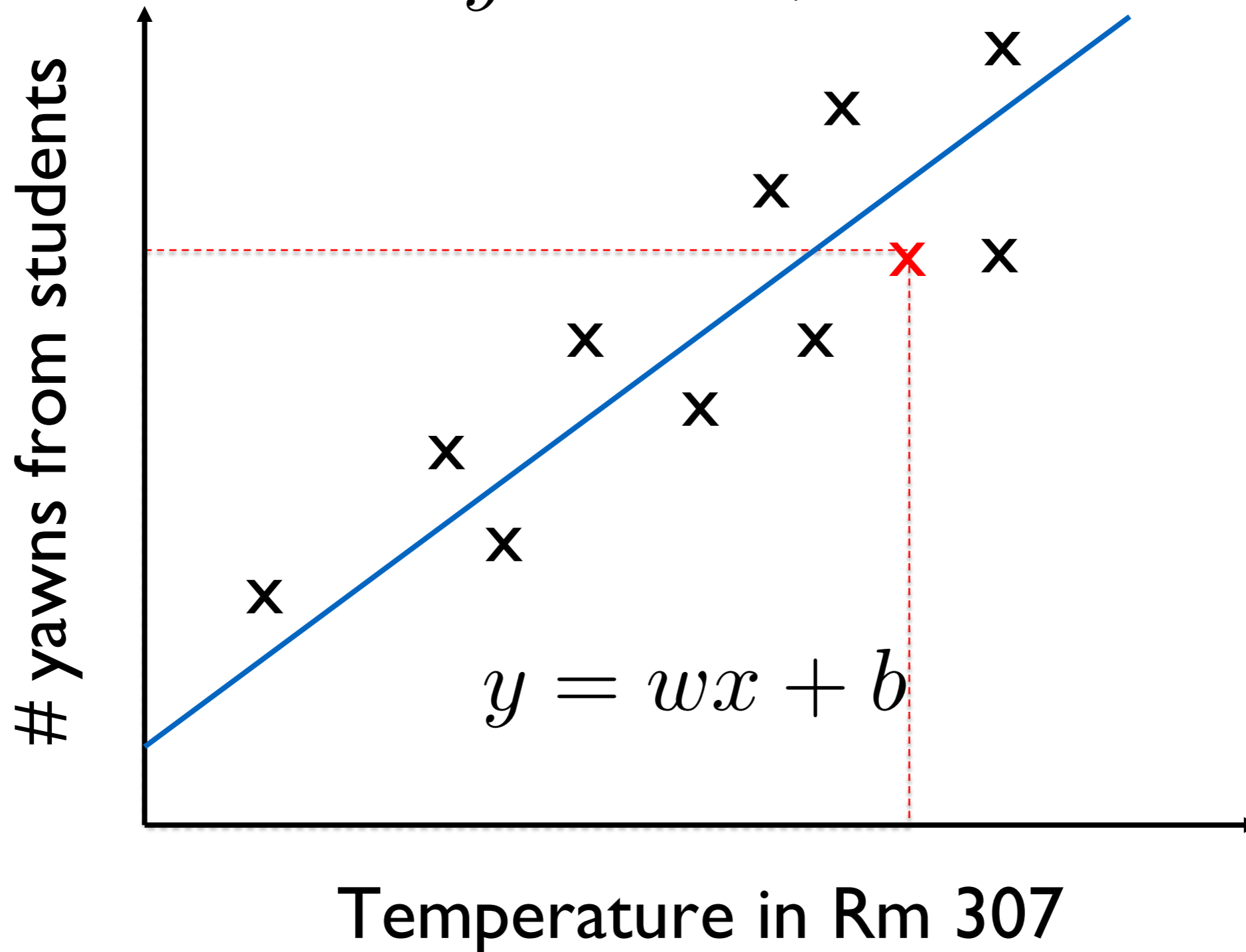
$$\sum_{i=1}^m (x^{(i)} - \bar{x})^2$$

It depends!

$$b = \bar{y} - w\bar{x}$$

Linear Regression: Prediction

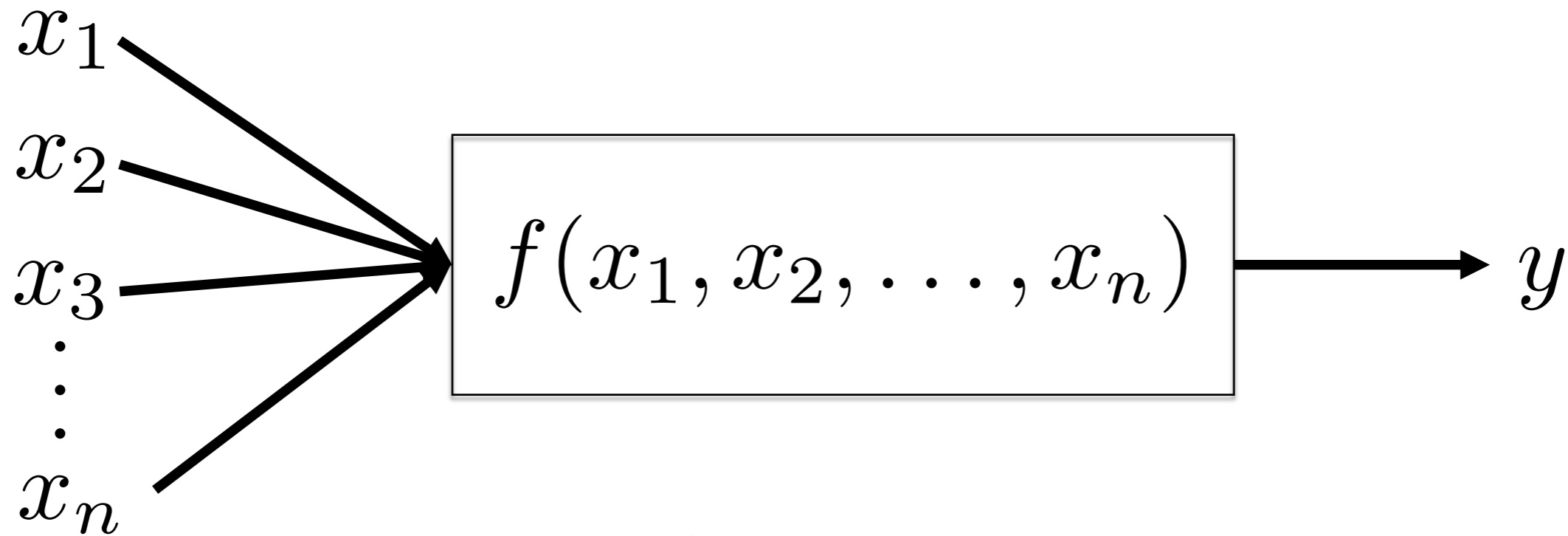
$$y = wx + b$$



Overview

- Philosophical questions
- Derivatives: What are they good for?
- Linear regression
- **Multiple linear regression**
- Logistic regression

Multiple Linear Regression



$$y = \sum_{j=1}^n (w_j x_j) + b$$

Multiple Linear Regression

Size (feet)	No. of bedrooms	No. of floors	Age (years)	Price (x\$1000)
2,350	5	2	45	500
1,600	3	2	20	450
2,000	3	2	30	250
854	2	1	10	200
560	1	1	30	180

Multiple Linear Regression: Training

- Given:

$$\{(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \dots, (x^{(m)}, y^{(m)})\}$$

- We want:

$$\hat{y}^{(i)} \approx y^{(i)}$$

Multiple Linear Regression: Training

- **Loss Function:** the discrepancy between the predicted and actual output values for a single training instance

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2} (y^{(i)} - \hat{y}^{(i)})^2$$

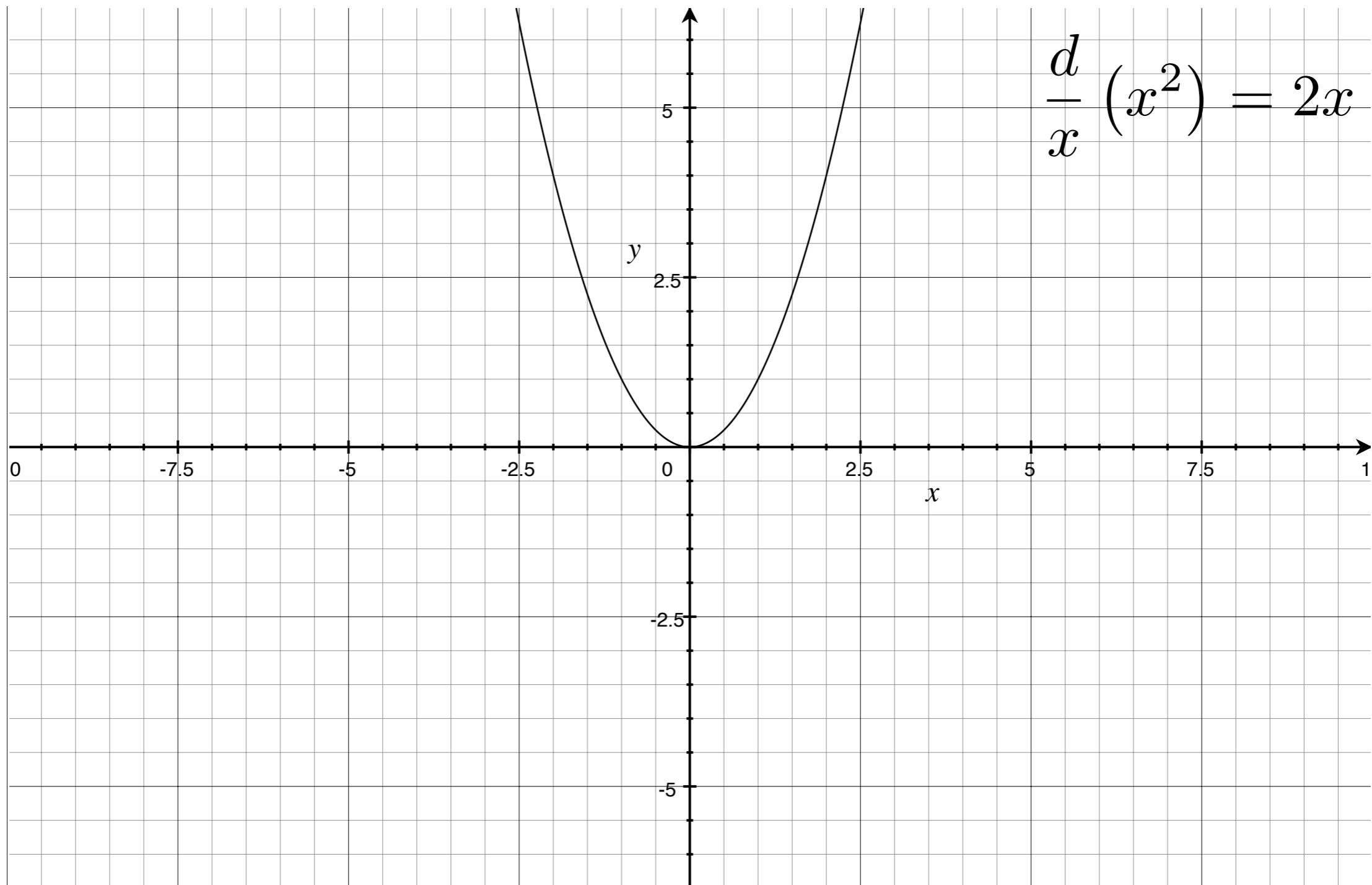
Multiple Linear Regression: Training

- **Cost Function:** the discrepancy between the predicted and actual output values for all training instances

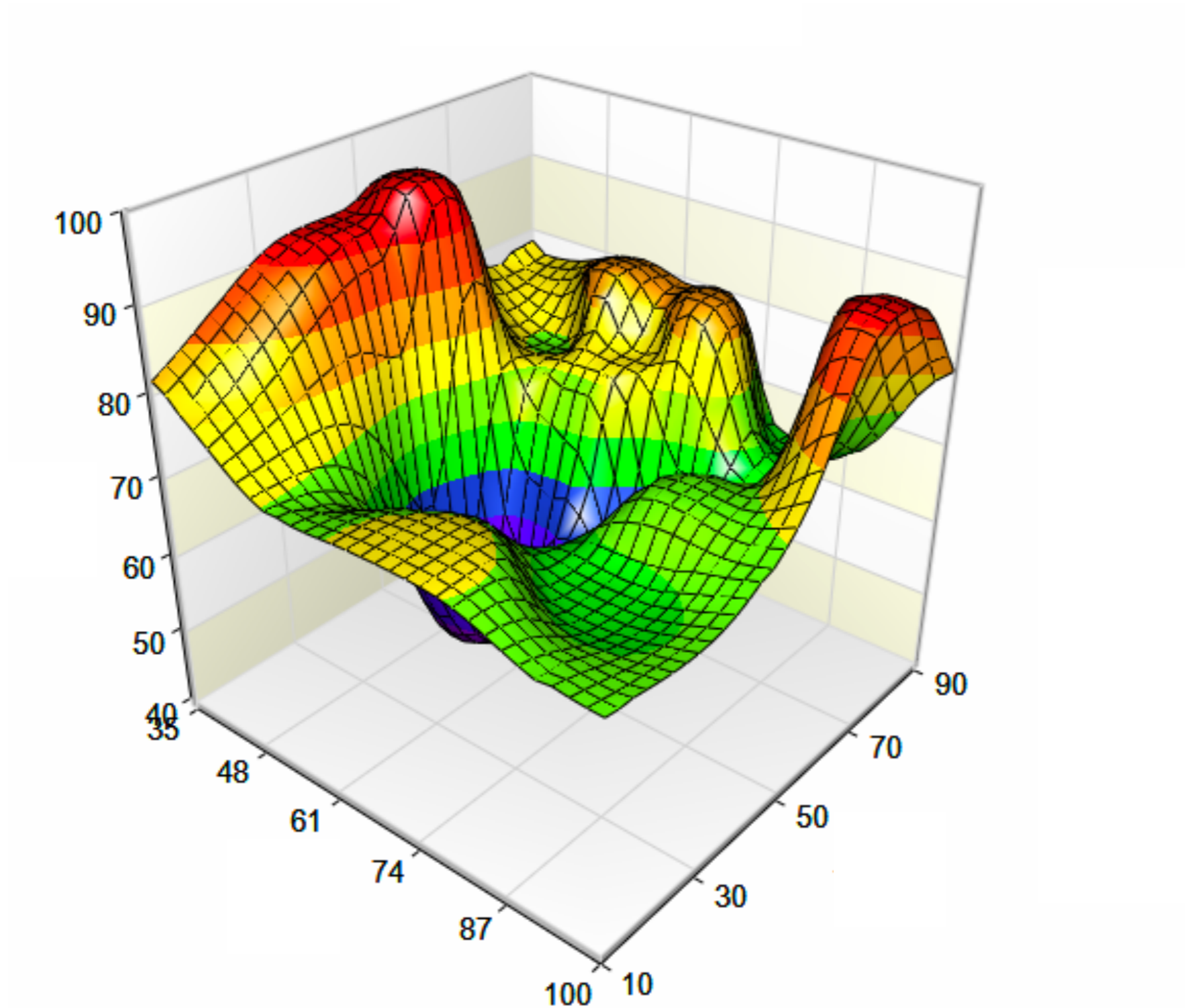
$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2} (y^{(i)} - \hat{y}^{(i)})^2$$

$$\mathcal{J}(w, b) = \frac{1}{m} \sum_{i=1}^m \left(\frac{1}{2} (y^{(i)} - \hat{y}^{(i)})^2 \right)$$

Gradient Descent: Intuition



Gradient Descent: Intuition



Gradient Descent

- **Loss Function:** the discrepancy between the predicted and actual output values for a single training instance

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2} (y^{(i)} - \hat{y}^{(i)})^2$$

- Let's see what the slope of the loss function is with respect to parameter b !
- **Note:** this will only consider one training example!

Gradient Descent

- Derivative of the **loss function** with respect to b

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2} (y^{(i)} - \hat{y}^{(i)})^2$$

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2} \left(y^{(i)} - \sum_{j=1}^n (w_j x_j^{(i)}) - b \right)^2$$

$$\frac{d}{db} \mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = - \left(y^{(i)} - \hat{y}^{(i)} \right)$$

$$\frac{d}{db} \mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \hat{y}^{(i)} - y^{(i)}$$

Gradient Descent

$$\frac{d}{db} \mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \hat{y}^{(i)} - y^{(i)}$$

Scenario	$\hat{y}^{(i)} - y^{(i)}$	Action!
$\hat{y}^{(i)} > y^{(i)}$	+	Decrease (nudge left)
$\hat{y}^{(i)} < y^{(i)}$	--	Increase (nudge right)
$\hat{y}^{(i)} \approx y^{(i)}$	0	Do nothing!

$$y = \sum_{j=1}^n (w_j x_j) + b$$

Gradient Descent

- **Loss Function:** the discrepancy between the predicted and actual output values for a single training instance

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2} (y^{(i)} - \hat{y}^{(i)})^2$$

- Let's see what the slope of the loss function is with respect to parameter w_j !
- **Note:** this will only consider one training example!

Gradient Descent

- Derivative of the **loss function** with respect to w_j

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2} (y^{(i)} - \hat{y}^{(i)})^2$$

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2} \left(y^{(i)} - \sum_{j=1}^n (w_j x_j^{(i)}) - b \right)^2$$

$$\frac{d}{dw_j} \mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = -x_j^{(i)} (y^{(i)} - \hat{y}^{(i)})$$

$$\frac{d}{dw_j} \mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = (\hat{y}^{(i)} - y^{(i)}) x_j$$

Gradient Descent

$$\frac{d}{dw_j} \mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = (\hat{y}^{(i)} - y^{(i)})x_j$$

Scenario	$\hat{y}^{(i)} - y^{(i)}$	Action!
$\hat{y}^{(i)} > y^{(i)}$	+	Go in the opposite direction as $x_j^{(i)}$
$\hat{y}^{(i)} < y^{(i)}$	-	Go in the same direction as $x_j^{(i)}$
$\hat{y}^{(i)} \approx y^{(i)}$	0	Do nothing!

$$y = \sum_{j=1}^n (w_j x_j) + b$$

Gradient Descent

- **Loss Function:** the discrepancy between the predicted and actual output values for a single training instance

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2} (y^{(i)} - \hat{y}^{(i)})^2$$

- Given one training example, we can take derivatives with respect to each parameter to see what direction we should be going to minimize the loss function.

Gradient Descent

- Repeat many times (or until convergence):

$$b \leftarrow b - \alpha \frac{1}{m} \sum_{i=1}^m \left(\hat{y}^{(i)} - y^{(i)} \right)$$

$$w_j \leftarrow w_j - \alpha \frac{1}{m} \sum_{i=1}^m \left(\left(\hat{y}^{(i)} - y^{(i)} \right) x_j^{(i)} \right)$$

Gradient Descent

- Repeat many times (or until convergence):

$$b \leftarrow b - \alpha \frac{1}{m} \sum_{i=1}^m \left(\hat{y}^{(i)} - y^{(i)} \right)$$

- If we are overshooting the target, reduce b
- If we are undershooting the target, increase b
- Otherwise, do nothing

Gradient Descent

- Repeat many times (or until convergence):

$$w_j \leftarrow w_j - \alpha \frac{1}{m} \sum_{i=1}^m \left((\hat{y}^{(i)} - y^{(i)}) x_j^{(i)} \right)$$

- If we are overshooting the target, reduce w_j proportional to the value of x_j
- If we are undershooting the target, increase w_j proportional to the value of x_j
- Otherwise, do nothing

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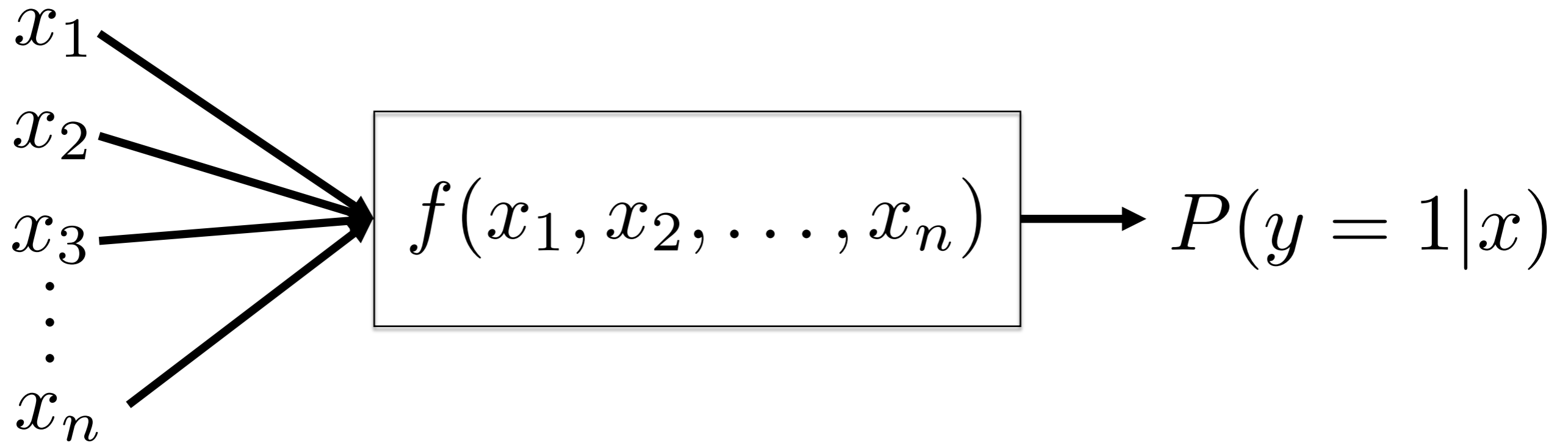
Logistic Regression

- Linear regression: predict y given x
- Multiple linear regression: predict y given x_1, x_2, \dots, x_n

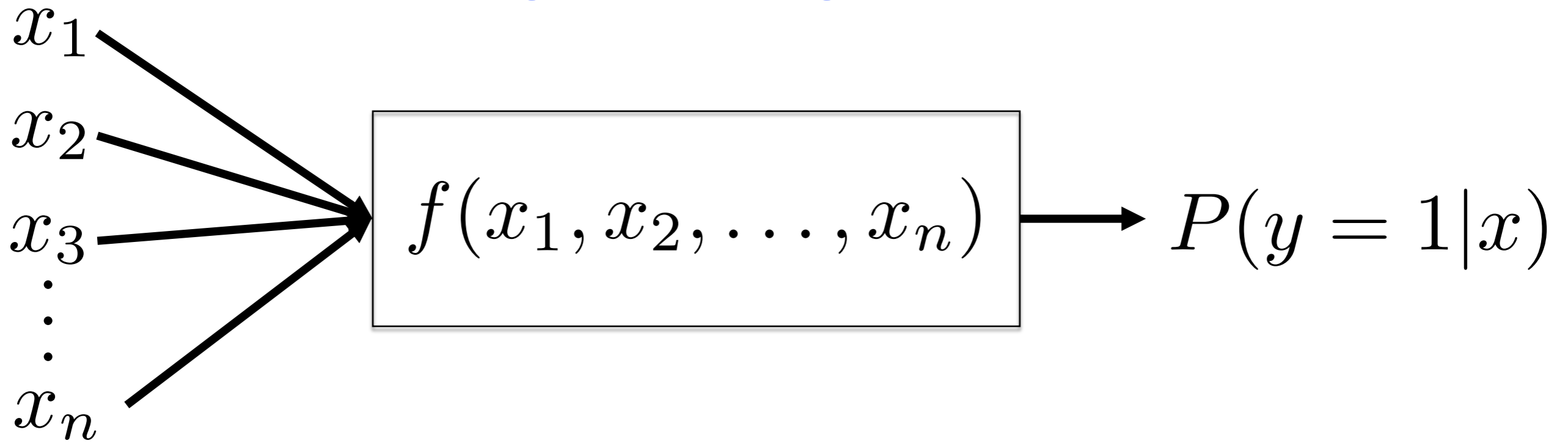
Logistic Regression

- **Logistic Regression:** predict $P(y=1 | x_1, x_2, \dots, x_n)$
- We can use logistic regression to do binary classification.

Logistic Regression



Logistic Regression

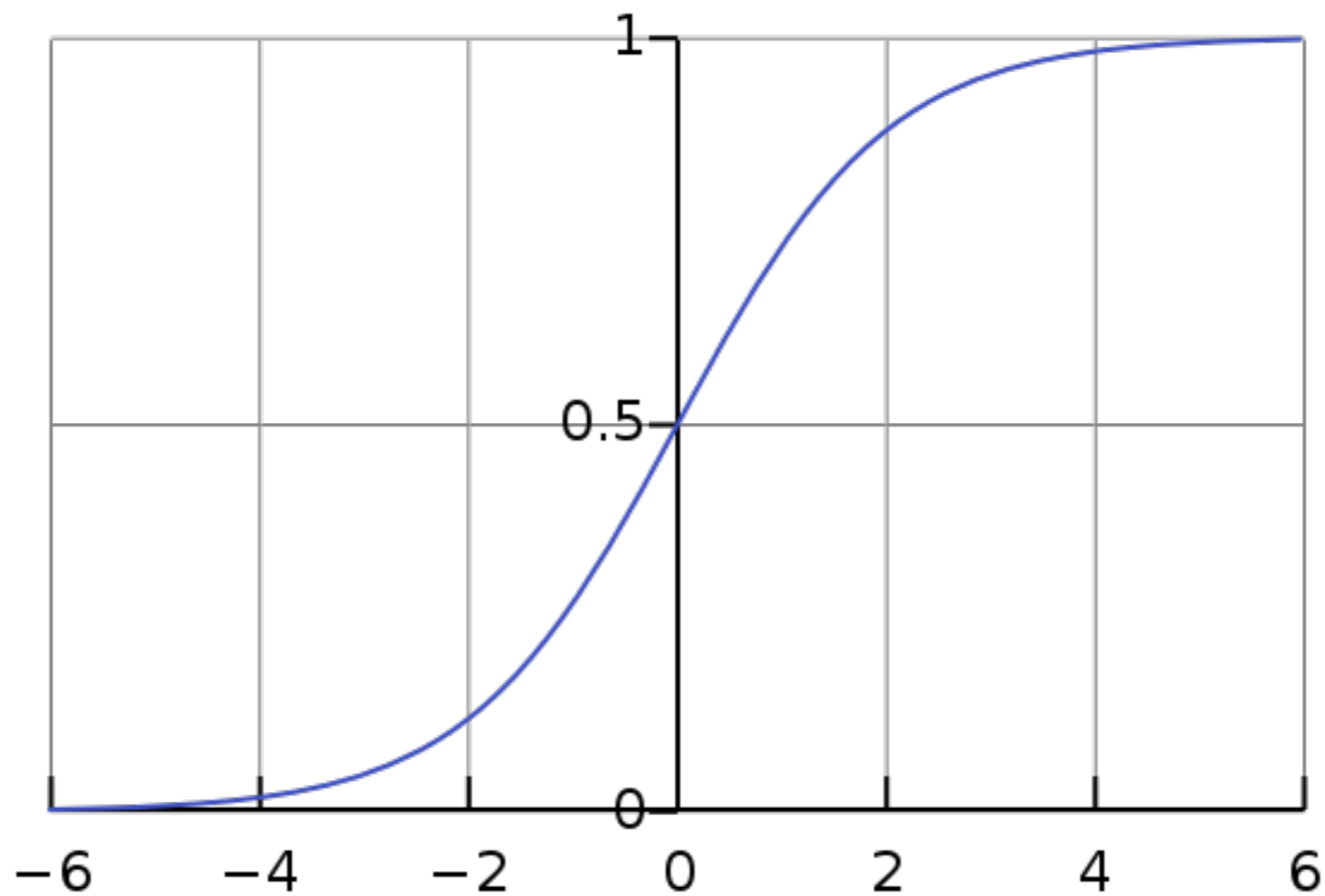


$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

$$z = \sum_{j=1}^n (w_j x_j) + b$$

Logistic Regression

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$



Logistic Regression

- **Loss Function:** the discrepancy between the predicted and actual output values for a single training instance

$$z = \sum_{j=1}^n (w_j x_j) + b$$

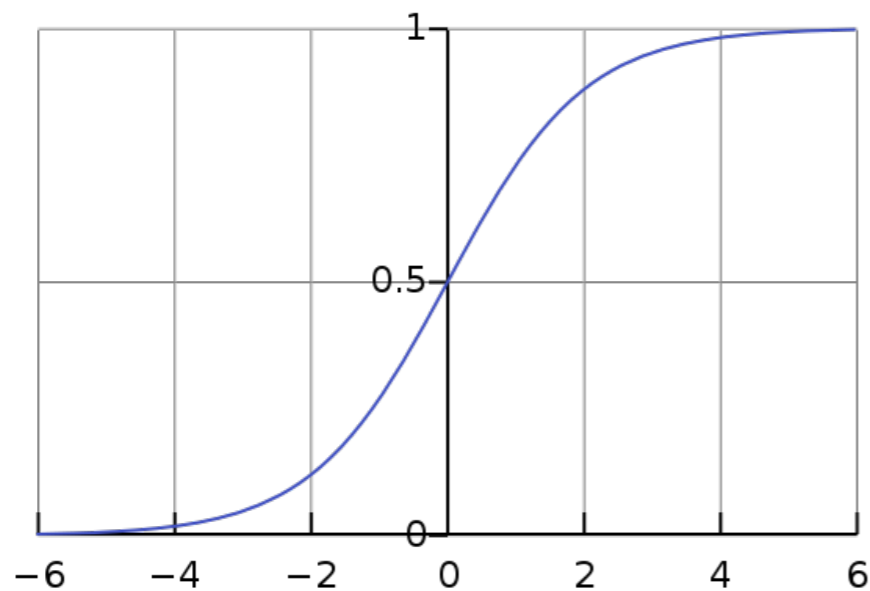
$$\hat{y} = \sigma(z) = \frac{1}{1 + e^{-z}}$$

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = - \left(y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log(1 - \hat{y}^{(i)}) \right)$$

Logistic Regression

- **Loss Function:** the discrepancy between the predicted and actual output values for a single training instance

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = - \left(y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log(1 - \hat{y}^{(i)}) \right)$$

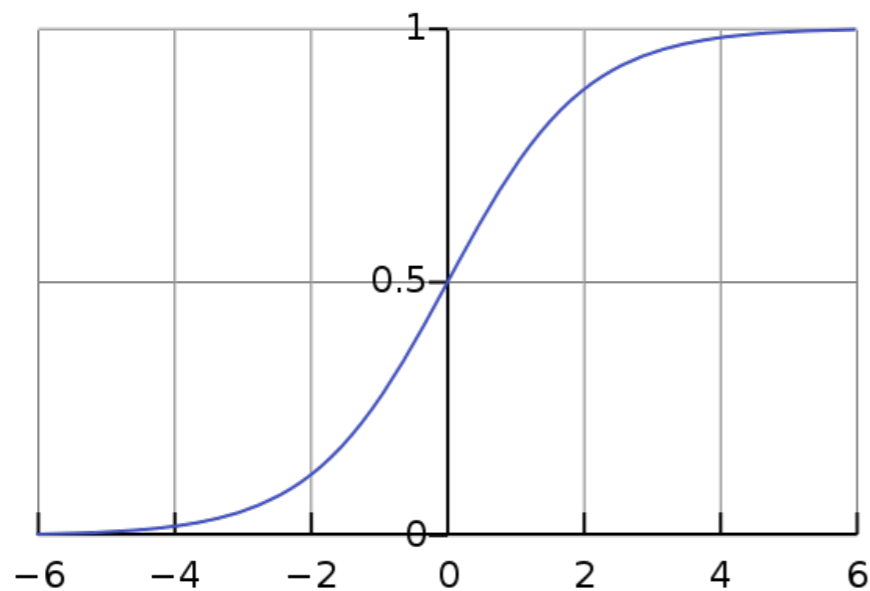


- If the true value is 1, we want the predicted value to be high.
- Remember: $\log(1) = 0$

Logistic Regression

- **Loss Function:** the discrepancy between the predicted and actual output values for a single training instance

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = - \left(y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log(1 - \hat{y}^{(i)}) \right)$$



- If the true value is 0, we want the predicted value to be low.
- Remember: $\log(1) = 0$

Logistic Regression

- **Loss Function:** the discrepancy between the predicted and actual output values for a single training instance

$$z = \sum_{j=1}^n (w_j x_j) + b$$

$$\hat{y} = \sigma(z) = \frac{1}{1 + e^{-z}}$$

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = - \left(y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log(1 - \hat{y}^{(i)}) \right)$$

$$\frac{d}{db} \mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \hat{y}^{(i)} - y^{(i)}$$

Logistic Regression

- **Loss Function:** the discrepancy between the predicted and actual output values for a single training instance

$$z = \sum_{j=1}^n (w_j x_j) + b$$

$$\hat{y} = \sigma(z) = \frac{1}{1 + e^{-z}}$$

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = - \left(y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log(1 - \hat{y}^{(i)}) \right)$$

$$\frac{d}{dw_j} \mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = (\hat{y}^{(i)} - y^{(i)}) x_j$$

Logistic Regression:

Gradient Descent

- Repeat many times (or until convergence):

$$b \leftarrow b - \alpha \frac{1}{m} \sum_{i=1}^m \left(\hat{y}^{(i)} - y^{(i)} \right)$$

$$w_j \leftarrow w_j - \alpha \frac{1}{m} \sum_{i=1}^m \left(\left(\hat{y}^{(i)} - y^{(i)} \right) x_j^{(i)} \right)$$

Logistic Regression:

Gradient Descent

- Repeat many times (or until convergence):

$$b \leftarrow b - \alpha \frac{1}{m} \sum_{i=1}^m \left(\hat{y}^{(i)} - y^{(i)} \right)$$

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Logistic Regression:

Gradient Descent

- Repeat many times (or until convergence):

$$w_j \leftarrow w_j - \alpha \frac{1}{m} \sum_{i=1}^m \left((\hat{y}^{(i)} - y^{(i)}) x_j^{(i)} \right)$$

- If we are overshooting the target, reduce w_j proportional to the value of x_j
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The Big Picture!

- Linear regression, multiple linear regression and logistic regression are examples of linear models
- Internally, linear models output a prediction based on a weighted combination of input features
- Features that are positively correlated with the target output get a high weight
- Features that are negatively correlated with the target output get a low weight
- Features that are uncorrelated with the target output get a zero weight